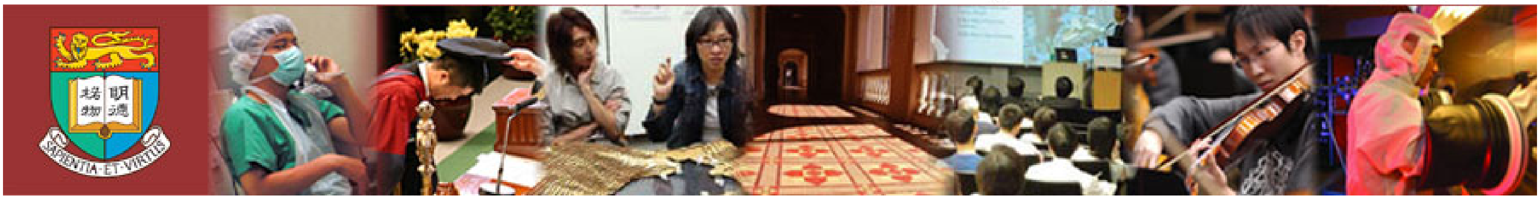

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Darboux covariant Lax pairs and infinite conservation laws of the (2+1)-dimensional breaking soliton equation

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In this paper, the binary Bell polynomials are applied to succinctly construct bilinear formulism, bilinear Bäcklund transformations, Lax pairs, and Darboux covariant Lax pairs for the (2+1)-dimensional breaking soliton equation. An extra auxiliary variable is introduced to get the bilinear formulism. The infinitely local conservation laws of the equation are found by virtue of its Lax equation and a generalized Miura transformation. All conserved densities and fluxes are given with explicit recursion formulas. © *2011 American Institute of Physics*. [doi:10.1063/1.3545804]

I. INTRODUCTION

As well-known, investigation of integrability for a nonlinear equation can be regarded as a pretest and the first step of its exact solvability. There are many significant properties, such as Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test, and bilinear Bäcklund transformation that can characterize integrability of nonlinear equations. Such work may pave a way for explicitly constructing their exact solutions in the future. Yet, the construction of bilinear Bäcklund transformation by using Hirota method is not as one would wish. It relies on a particular skill in using appropriate exchange formulas which are connected with the linear representation of the system.[1–4] However, in recent years Lambert and co-workers have proposed a procedure to obtain parameter families of bilinear Bäcklund transformation for soliton equations in a lucid and systematic way based on the use of Bell polynomials.[5–7] The Bell polynomials are found to play an important role in the characterization of bilinearizable equations. As a consequence, bilinear Bäcklund transformation with single field can be linearized into corresponding Lax pairs. Their method provides a short way to bilinear Bäcklund transformation and Lax pairs of nonlinear equations, which establishes a deep relation between integrability of a nonlinear equation and the Bell polynomials.

In this paper, we extend the binary Bell polynomial approach to construct bilinear formulism, bilinear Bäcklund transformations, Lax pairs, and Darboux covariant Lax pairs of the following (2+1)-dimensional breaking soliton equation

$$u_{xt} + u_{xxxy} - 4u_x u_{xy} - 2u_{xx}u_y = 0, \tag{1.1}$$

which was first presented by Calogero and Degasperis.[8,9] The similar equation

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 4u_{xx}u_y = 0 \tag{1.2}$$

was studied by Bogoyavlenskii, where overlapping solutions were generated.[10,11] Equations (1.1) and (1.2) are typical so-called breaking soliton equations to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. In recent years, a large number of papers have been focusing on Painlevé property, dromion-like structures, and various exact solutions of these two equations.[12–26] But their integrability, to the best of our

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knowledge, have not been studied in detail except to previous little work, as seen, e.g., in Refs. 8–10 and 27–30. In addition, the study on conservation laws of (2+1)-dimensional equations has been still less in contrast with the (1+1)-dimensional case. The existence of infinitely local conservation laws can be considered as one of the many remarkable properties that deemed to characterize soliton equations. The more conservation laws one finds the closer one gets to the complete solution. A conservation law of a higher dimensional system is, by definition, an equation in divergence form

$$\mathrm{div}\mathbf{F} = \partial \mathbf{x_1 F_1} + \cdots + \partial \mathbf{x_m F_m} = \mathbf{0}, \tag{1.3}$$

where vector function $\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u_x}, \ldots) = (\mathbf{F_1}, \ldots, \mathbf{F_m})$ depending on $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_m})$, $u = u(\mathbf{x})$ and its derivations. The vector function $\mathbf{F}$ is called conserved flux since Eq. (1.3) implies that a net flow of F through any $(m-1)$-dimensional closed surface Ω in the m-dimensional space $\partial\Omega$ is zero according to the Gauss formula, namely

$$\int_{\Omega} \mathbf{F} \cdot \mathbf{ndx} = \int_{\partial\Omega} \mathrm{div}\mathbf{FdS} = \mathbf{0}.$$

Physically a conservation law means that the rate of change of F_1inside any spatial domain must equal the $(F_2, \cdots, F_m)$ through the surface of the domain.

Here we shall employ binary Bell polynomials to systematically construct bilinear representation, Bäcklund transformation, Lax pair, and Darboux covariant Lax pair of Eq. (1.1). The infinitely local conservation laws of the equation will be constructed through its Lax equation and a generalized Miura transformation. In Sec. II, we briefly present necessary notations on multidimensional binary Bell polynomials. These results will then be applied to construct the bilinear representation, Bäcklund transformation, Lax pair, Darboux covariant Lax pair, and infinite conservation laws to Eq. (1.1) in Secs. III– VI, respectively.

II. MULTIDIMENSIONAL BINARY BELL POLYNOMIALS

The main tool used in this paper is a class of generalized multidimensional binary Bell polynomials. To make our presentation easy to understand and self-contained, we first fix some necessary notations on the Bell polynomials, for details refer, for instance, to Lambert and Gilson's work.[5–7]

Let us start with original standard Bell polynomials.

Definition 1: Assume that $r > 0$ denote a constant integer; $n \geq 0$ denote an arbitrary integer; and x, t denote independent variables; then the polynomials about variables x and t

$$\xi_n(x, t) \equiv \exp(-tx^r)\partial_x^n \exp(tx^r) \tag{2.1}$$

is called classical Bell polynomials or Hermite-Bell polynomials, which was originally introduced by Bell.[30]

For $r = 2$, the Bell polynomials $\xi_n(x, t)$ is exactly Hermite polynomials. The first few lowest order Bell Polynomials are

$$\xi_0(x, t) = 1, \quad \xi_1(x, t) = rtx^{r-1}, \xi_2(x, t) = r^2t^2x^{2r-2} + r(r-1)tx^{r-2},$$
$$\xi_3(x, t) = r^3t^3x^{3r-3} + 3r^2(r-1)t^2x^{2r-3} + r(r-1)(r-2)tx^{r-3}.$$

In general,

$$\xi_n(x, t) = n! \sum_{j=a}^{n} \frac{t^j x^{rj-n}}{j!} \sum_{l=0}^{b} (-1)^l \binom{j}{l} \binom{r(j-l)}{n},$$

where $a = n - [n(r-1)/r]$ and $b = [(rh - n)/r]$ denote the greatest integer in their brackets.

Next, we consider general generalization of the Bell polynomials (2.1).

Definition 2: Let $n_k \geq 0,\ k = 1, \cdots, \ell$ denote arbitrary integers, $f = f(x_1, \cdots, x_\ell)$ be a C^∞ multivariable function, then

$$Y_{n_1x_1,\cdots,n_\ell x_\ell}(f) \equiv \exp(-f)\partial_{x_1}^{n_1} \cdots \partial_{x_\ell}^{n_\ell} \exp(f) \tag{2.2}$$

are polynomials in the partial derivatives of f with respect to $x_1, \cdots, x_\ell$, which we call multidimensional Bell polynomials (generalized Bell polynomials or Y-polynomials).

For the special case $f = f(x,t)$, the associated two-dimensional Bell polynomials defined by (2.2) read

$$Y_x(f) = f_x,\ Y_{2x}(f) = f_{2x} + f_x^2, Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3,$$
$$Y_{x,t}(f) = f_{x,t} + f_x f_t, Y_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2f_{x,t} f_x + f_x^2 f_t, \cdots.$$

For the special case $n_1 = n,\ n_2 = 0,\ f = f(x,t) = tx^r$ with the constant integer $r > 0$, then the multidimensional Bell polynomials (2.2) exactly reduces the classical Bell polynomials (2.1)

$$Y_{nx}(f) = \exp(-tx^r)\partial_x^n \exp(tx^r) = \xi_n(x,t).$$

This implies that the multidimensional Bell polynomials (2.2) is a generalization of the classical Bell polynomials (2.1).

Definition 3: Based on the use of above Bell polynomials (2.2), the multidimensional binary Bell polynomials ($\mathcal{Y}$-polynomials) can be defined as follows:

$$\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w) = Y_{n_1x_1,\cdots,n_\ell x_\ell}(f)\,\Big|_{f_{r_1x_1,\cdots,r_\ell x_\ell}=\begin{cases} v_{r_1x_1,\cdots,r_\ell x_\ell}, & r_1 + \cdots + r_\ell \quad \text{is odd}, \\ w_{r_1x_1,\cdots,r_\ell x_\ell}, & r_1 + \cdots + r_\ell \quad \text{is even}, \end{cases}}$$

which is a multivariable polynomials with respect to all partial derivatives $v_{r_1x_1,\cdots,r_\ell x_\ell}$ ($r_1 + \cdots + r_\ell$ odd) and $w_{r_1x_1,\cdots,r_\ell x_\ell}$ ($r_1 + \cdots + r_\ell$ even), $r_k = 0, \cdots, n_k,\ k = 0, \cdots, \ell$.

The binary Bell polynomials also inherits the easily recognizable partial structure of the Bell polynomials. The first few lowest order binary Bell Polynomials are

$$\mathcal{Y}_x(v) = v_x,\ \mathcal{Y}_{2x}(v,w) = w_{2x} + v_x^2, \mathcal{Y}_{x,t}(v,w) = w_{xt} + v_x v_t.$$
$$\mathcal{Y}_{3x}(v,w) = v_{3x} + 3v_x w_{2x} + v_x^3, \ldots. \tag{2.3}$$

Theorem 1: *(Ref. 5) The link between binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w)$ and the standard Hirota bilinear equation $D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell} F \cdot G$ can be given by an identity*

$$\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v = \ln F/G, w = \ln FG) = (FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell} F \cdot G, \tag{2.4}$$

in which $n_1 + n_2 + \cdots + n_\ell \geq 1$, and operators $D_{x_1}, \cdots, D_{x_\ell}$ are classical Hirota's bilinear operators defined by

$$D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell} F \cdot G = (\partial_{x_1} - \partial_{x_1'})^{n_1} \cdots (\partial_{x_\ell} - \partial_{x_\ell'})^{n_\ell} F(x_1, \cdots, x_\ell) G(x_1', \cdots, x_\ell')|_{x_1'=x_1,\cdots,x_\ell'=x_\ell}.$$

In the particular case when $F = G$, the formula (2.4) becomes

$$F^{-2} D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell} G \cdot G = \mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(0, q = 2\ln G)$$
$$= \begin{cases} 0, & n_1 + \cdots + n_\ell \quad \text{is odd}, \\ P_{n_1x_1,\cdots,n_\ell x_\ell}(q), & n_1 + \cdots + n_\ell \quad \text{is even}, \end{cases} \tag{2.5}$$

in which the P-polynomials can be characterized by an equally recognizable even part partitional structure

$$P_{2x}(q) = q_{2x},\ P_{x,t}(q) = q_{xt},\ P_{4x}(q) = q_{4x} + 3q_{2x}^2, P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3, \ldots. \tag{2.6}$$

The formulae (2.6) and (2.5) will prove particularly useful in connecting nonlinear equations with their corresponding bilinear equations. This means that once a nonlinear equation is expressible as a linear combination of P-polynomials, then it can be transformed into a linear equation.

Theorem 2: *(Ref. 5) The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w)$ can be separated into P-polynomials and Y-polynomials*

$$\begin{aligned}(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}F\cdot G &= \mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w)|_{v=\ln FG, w=\ln FG}\\ &= \mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,v+q,)|_{v=\ln F/G, q=2\ln G}\\ &= \sum_{n_1+\cdots+n_\ell=even}\sum_{r_1=0}^{n_1}\cdots\sum_{r_\ell=0}^{n_\ell}\prod_{i=1}^{\ell}\binom{n_i}{r_i}P_{r_1x_1,\cdots,r_\ell x_\ell}(q)Y_{(n_1-r_1)x_1,\cdots,(n_\ell-r_\ell)x_\ell}(v).\end{aligned} \tag{2.7}$$

The key property of the multidimensional Bell polynomials

$$Y_{n_1x_1,\cdots,n_\ell x_\ell}(v)|_{v=\ln\psi} = \psi_{n_1x_1,\cdots,n_\ell x_\ell}/\psi, \tag{2.8}$$

implies that the binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w)$ can still be linearized by means of the Hopf-Cole transformation $v = \ln\psi$, that is, $\psi = F/G$. The formulae (2.7) and (2.8) will then provide the shortest way to the associated Lax system of nonlinear equations.

III. BILINEAR REPRESENTATION

In this section, we will see that an extra auxiliary variable is asked to get bilinear representation of Eq. (1.1), which is more difficult than Eq. (1.2).[24] In order to detect the existence of linearizable representation of Eq. (1.1), we introduce a potential field q by setting

$$u = cq_x \tag{3.1}$$

with c being free constant to be the appropriate choice such that Eq. (1.1) connect with P-polynomials. Substituting transformation (3.1) into Eq. (1.1), we can write the resulting equation in the form

$$q_{2x,t} + \frac{2}{3}q_{4x,y} - 2c(q_{2x}q_{2x,y} + q_{xy}q_{3x}) + \frac{1}{3}q_{4x,y} - 2cq_{2x}q_{2x,y} = 0, \tag{3.2}$$

where we will see that such decomposition is necessary to get bilinear form of Eq. (1.1). Further integrating Eq. (3.2) with respect to x yields

$$E(q) \equiv q_{x,t} + \frac{2}{3}(q_{3x,y} + 3q_{2x}q_{xy}) + \frac{1}{3}\partial_x^{-1}\partial_y(q_{4x} + 3q_{2x}^2) = 0, \tag{3.3}$$

if we set $c = -1$ according to the formula (2.4).

In order to write Eq. (3.3) in local bilinear form, we should eliminate effect of the integration ∂_x^{-1}. To this end, we introduce an auxiliary variable z and impose a subsidiary constraint condition

$$q_{4x} + 3q_{2x}^2 + q_{xz} = 0, \tag{3.4}$$

on account of which, Eq. (3.3) becomes

$$q_{x,t} + \frac{2}{3}(q_{3x,y} + 3q_{2x}q_{xy}) - \frac{1}{3}q_{yz} = 0. \tag{3.5}$$

Now according to the formula (2.4), Eqs. (3.3) and (3.4) are then cast into a pair of equations in the form of P-polynomials

$$P_{4x}(q) + P_{xz}(q) = 0,\ P_{x,t}(q) + \frac{2}{3}P_{3x,y}(q) - \frac{1}{3}P_{yz}(q) = 0. \tag{3.6}$$

Finally, according to the property (2.3), under the change of dependent variable

$$q = 2\ln G \iff u = cq_x = -2(\ln G)_x,$$

Eq. (3.6) produces the bilinear representation of the breaking soliton equation (1.1) as follows:

$$(D_x^4 + D_x D_z)G \cdot G = 0, (D_x D_t + \frac{2}{3} D_y D_x^3 - \frac{1}{3} D_y D_z)G \cdot G = 0. \tag{3.7}$$

This equation is easy to be solved for multisoliton solutions by using Hirota's bilinear method. For example, the regular one-soliton-like solution reads

$$u = k \tanh \frac{kx + 3ly + lk^2 t}{2},$$

where k and l are two constants. The multisoliton solution are omitted here since exactly solving Eq. (1.1) is not our main purpose in this paper.

IV. BÄCKLUND TRANSFORMATION AND LAX PAIR

Next, we search for the bilinear Bäcklund transformation and Lax pair of the breaking soliton equation (1.1). Let

$$q = 2 \ln G, \quad q' = 2 \ln F$$

be two different solutions of Eq. (3.3), respectively. On introducing two new variables

$$w = (q' + q)/2 = \ln(FG), \quad v = (q' - q)/2 = \ln(F/G), \tag{4.1}$$

we associate the two-field condition

$$\begin{aligned} &E(q') - E(q) = E(w + v) - E(w - v) \\ &= 2v_{xt} + 2v_{3x,y} + 4w_{2x}v_{x,y} + 4w_{x,y}v_{2x} + 4\partial_x^{-1}(w_{2x}v_{2x,y} + w_{2x,y}v_{2x}) \\ &= 2\partial_x[\mathcal{Y}_t(v) + \mathcal{Y}_{2x,y}(v, w)] + R(v, w) = 0 \end{aligned} \tag{4.2}$$

with

$$R(v, w) = -2\partial_x[(w_{2x} + v_x^2)v_y] + 4w_{2x}v_{xy} - 4w_{2x,y}v_x + 4\partial_x^{-1}(w_{2x}v_{2x,y} + w_{2x,y}v_{2x}).$$

This two-field condition can be regarded as the natural ansatz for a bilinear Bäcklund transformation and may produce the required transformation under appropriate additional constraints.

In order to decouple the two-field condition (4.2) into a pair of constraints, we impose such a constraint which enable us to express $R(v, w)$ as the form of x-derivative of $\mathcal{Y}$-polynomials. The simplest possible choice of such constraint may be

$$\mathcal{Y}_{2x}(v, w) = w_{2x} + v_x^2 = \lambda, \tag{4.3}$$

on account of which, directly computing the $R(v, w)$, we find that

$$R(v, w) = 2\lambda v_{xy} + 4w_{2x}v_{xy} - 4w_{2x,y}v_x - 4v_x^2 v_{xy} = 6\lambda v_{xy}, \tag{4.4}$$

where we have used the relations $w_{2x,y} = -2v_x v_{xy}$ and $w_{2x} = \lambda - v_x^2$.

Then, combining the relations (4.2)–(4.4), we deduce a coupled system of $\mathcal{Y}$-polynomials

$$\begin{aligned} &\mathcal{Y}_{2x}(v, w) - \lambda = 0, \\ &\partial_x \mathcal{Y}_t(v) + \partial_x[\mathcal{Y}_{2x,y}(v, w) + 3\lambda \mathcal{Y}_y(v)] = 0, \end{aligned} \tag{4.5}$$

where we prefer the second equation in the conserved form without integration with respect to x, which is useful to construct conservation laws later. By application of the identity (2.4), the system (4.5) immediately leads to the bilinear Bäcklund transformation

$$\begin{aligned} &(D_x^2 - \lambda)F \cdot G = 0, \\ &(D_t + D_y D_x^2 + 3\lambda D_y - \mu)F \cdot G = 0, \end{aligned}$$

where we have integrated the second equation in the system (4.5) with respect to x, and $\mu = \mu(t)$ is an arbitrary function.

By transformation $v = \ln\psi$, it follows from the formulae (2.7) and (2.8) that

$$\mathcal{Y}_t(v) = \psi_t/\psi,\ \mathcal{Y}_y(v) = \psi_y/\psi, \mathcal{Y}_{2x}(v, w) = q_{2x} + \psi_{2x}/\psi,$$
$$\mathcal{Y}_{2x,y}(v, w) = 2q_{xy}\psi_x/\psi + q_{2x}\psi_y/\psi + \psi_{2x,y}/\psi,$$

on account of which, the system (4.5) is then linearized into a system with double parameters λ and μ

$$L_1\psi \equiv (\partial_x^2 + q_{2x})\psi = \lambda\psi, \tag{4.6}$$

$$\psi_t + L_2\psi \equiv [\partial_t + \partial_y\partial_x^2 + 2q_{xy}\partial_x + (q_{2x} + 3\lambda)\partial_y]\psi = \mu\psi, \tag{4.7}$$

or equivalently,

$$L_1\psi = (\partial_x^2 + q_{2x})\psi = \lambda\psi,$$
$$\psi_t + L_2\psi = (\partial_t + 2q_{xy}\partial_x + 4\lambda\partial_y + \lambda_y - q_{2x,y})\psi = \mu\psi,$$

where we have used Eq. (4.6) to get the second equation, and allow the y and t dependence of λ.

It is easy to check that, for the following equations:

$$L_1\psi = \lambda\psi,\ \ \psi_t + Ł_2\psi = \mu\psi,\ \ \lambda_t \equiv f(\lambda) = -4\lambda\lambda_y, \tag{4.8}$$

their integrability condition

$$\begin{aligned} 0 &= L_{1,t} - f(L_1) - [L_1, L_2] \\ &= -(\lambda_t + 4\lambda\lambda_y) + q_{2x,t} + q_{4x,y} + 4q_{2x}q_{2x,y} + q_{3x}q_{xy}, \end{aligned} \tag{4.9}$$

exactly gives the breaking soliton equation (1.1) by replacing $-q_x$ by u and using the nonisospectral condition

$$\lambda_t + 4\lambda\lambda_y = 0. \tag{4.10}$$

Starting from the Lax pair (4.8), the Darboux transformation and soliton-like solutions of the breaking soliton equation (1.1) can be established, here we omit them without consideration.

V. DARBOUX COVARIANT LAX PAIR

In this section, based on the assumption that the parameter λ is independent of variables x, y, and t, we present a kind of Darboux covariant Lax pair whose form is invariant under a certain gauge transformation. Let us go back to the breaking soliton equation (1.1) and the associated Lax pair (4.6)–(4.7). Suppose that ϕ is a solution eigenvalue equation (4.6). It is well-known that the gauge transformation

$$T = \phi\partial_x\phi^{-1} = \partial_x - \sigma,\ \ \sigma = \partial_x \ln\phi \tag{5.1}$$

map the operator $L_1(q) - \lambda$ onto a similar operator

$$T(L_1(q) - \lambda)T^{-1} = \tilde{L}_1(\tilde{q}) - \lambda,$$

which satisfies the covariance condition

$$\tilde{L}_1(\tilde{q}) = L_1(\tilde{q} = q + \Delta q)\ \text{ with }\ \Delta q = 2\ln\phi.$$

But it can be verified that similar property does not hold for the evolution equation (4.7).

Next step is to find another third order operator $L_{2,\text{cov}}(q)$ with appropriate coefficients, such that $\partial_t + L_{2,\text{cov}}(q)$ be mapped, by gauge transformation (5.1), onto a similar operator $\tilde{L}_{2,\text{cov}}(\tilde{q})$ which satisfies the covariance condition

$$\tilde{L}_{2,\text{cov}}(\tilde{q}) = L_{2,\text{cov}}(\tilde{q} = q + \Delta q).$$

Suppose that ϕ is a solution of the following Lax pair

$$L_1\phi = \lambda\phi, \quad \phi_t + L_{2,\text{cov}}\phi = 0, \quad L_{2,\text{cov}} = 4\partial_y\partial_x^2 + b_1\partial_x + b_2\partial_y + b_3, \tag{5.2}$$

where $b_1,\ b_2$, and b_3 are functions to be determined. It suffices that we require the transformation T map the operator $\partial_t + L_{2,\text{cov}}$ onto the similar one

$$T(\partial_t + L_{2,\text{cov}})T^{-1} = \partial_t + \tilde{L}_{2,\text{cov}}, \quad \tilde{L}_{2,\text{cov}} = 4\partial_y\partial_x^2 + \tilde{b}_1\partial_x + \tilde{b}_2\partial_y + \tilde{b}_3, \tag{5.3}$$

where $\tilde{b}_1, \tilde{b}_2$, and $\tilde{b}_3$ satisfy the covariant condition

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2, 3. \tag{5.4}$$

It follows from (5.2) and (5.3) that

$$\Delta b_1 = \tilde{b}_1 - b_1 = 4\sigma_y, \quad \Delta b_2 = \tilde{b}_2 - b_2 = 8\sigma_x, \tag{5.5}$$

$$\Delta b_3 = \tilde{b}_3 - b_3 = \sigma\Delta b_1 + 8\sigma_{xy} + b_{1,x}, \tag{5.6}$$

and σ satisfies

$$\sigma_t + 4\sigma_{2x,y} + \tilde{b}_1\sigma_x + \tilde{b}_2\sigma_y + \sigma\Delta b_3 + b_{3,x} = 0. \tag{5.7}$$

According to the relation (5.4), it remains to determine $b_1,\ b_2$, and b_3 in the form of polynomial expressions in terms of derivatives of q

$$b_j = F_j(q, q_x, q_y, q_{xy}, q_{2x}, q_{2y}, q_{2x,y}, \ldots), \quad j = 1, 2, 3$$

such that

$$\Delta F_j = F_j(q + \Delta q, q_x + \Delta q_x, q_y + \Delta q_y, \cdots) - F_j(q, q_x, q_y, \cdots) = \Delta b_j \tag{5.8}$$

with $\Delta q_{kx,ly} = 2(\ln q)_{kx,ly},\ k, l = 1, 2, \cdots$, and the Δb_j being determined by the relations (5.5)–(5.7).

Expanding the left hand of Eq. (5.8), we obtain

$$\Delta b_1 = \Delta F_1 = F_{1,q}\Delta q + F_{1,q_x}\Delta q_x + F_{1,q_y}\Delta q_y + F_{1,q_{xy}}\Delta q_{xy} + \cdots = 4\sigma_y = 2\Delta q_{xy},$$

which implies that we can determine b_1 up to a arbitrary constant c_1, namely,

$$b_1 = F_1(q_{xy}) = 2q_{xy} + c_1. \tag{5.9}$$

Proceeding in the same way, we deduce the function b_2 as follows

$$b_2 = F_2(q_{2x}) = 4q_{2x} + c_2 \tag{5.10}$$

with c_2 being arbitrary constant.

We see from the relation (5.6) that Δb_3 contain the term $b_{1,x} = q_{2x,y}$, which should be eliminated such that Δb_3 admits the form (5.8). By means of the eigenvalue equation in (5.2), we can find the following relation

$$q_{2x,y} = -\sigma_{xy} - 2\sigma\sigma_y. \tag{5.11}$$

Substituting (5.9) and (5.11) into (5.6) yields

$$\Delta b_3 = 4\sigma\sigma_y + 8\sigma_{xy} + 2q_{2x,y} = 6\sigma_{xy} = 3\Delta q_{2x,y}.$$

It is can verified that the third condition

$$\Delta F_3 = F_{3,q}\Delta q + F_{3,q_x}\Delta q_x + F_{3,q_y}\Delta q_y \cdots = \Delta b_3$$

can be satisfied, if one chooses

$$b_3 = F_3(q_{2x,y}) = 3q_{2x,y} + c_3, \tag{5.12}$$

in which c_3 is arbitrary constant.

Setting $c_1 = c_2 = 0,\ c_3 = -\mu$ in (5.9), (5.10), and (5.12), it follows from (5.2) that we find the following Darboux covariant evolution equation

$$\phi_t + L_{2,\mathrm{cov}}\phi = 0, \quad L_{2,\mathrm{cov}} = 4\partial_y\partial_x^2 + 2q_{xy}\partial_x + 4q_{2x}\partial_y + 3q_{2x,y} - \mu,$$

which is in agreement with Eq. (5.7). Moreover, the relation between the operator $L_{2,\mathrm{cov}}$ and the operator L_2 is given by

$$L_{2,\mathrm{cov}} = L_2 + 3\partial_x(L_1 - \lambda).$$

Under nonisospectral condition $\lambda_t + 4\lambda\lambda_y = 0$, the integrability condition of the Darboux covariant Lax pair (5.2) precisely give rise to Eq. (1.1) in Lax representation

$$[\partial_t + L_{2,\mathrm{cov}}, L_1] = -(u_{xt} + u_{3x,y} - 4u_x u_{x,y} - 2u_{2x}u_y).$$

In a similar way step by step, we can obtain higher operators, which are Darboux covariant with respect to L_1, so as to produce higher order members of the breaking soliton hierarchy.

VI. INFINITE CONSERVATION LAWS

In this section, we derive the infinitely local conservation laws for breaking soliton equation (1.1) through the Lax equation (4.9) and a generalized Miura transformation.

Let us first see the role of the nonisospectral parameter $\lambda = \lambda(y, t)$ in the Lax equation. In fact, the nonisospectral condition (4.10) is a conservation law

$$\lambda_t + (2\lambda^2)_y = 0, \tag{6.1}$$

which implies that for any domain Ω with the boundary $\partial\Omega$ in space R^2, the following equation hold

$$\frac{\partial}{\partial t}\int_{\Omega} \lambda^k dxdy = -\frac{4k}{k+1}\int_{\partial\Omega}\frac{\partial\lambda^{k+1}}{\partial y}dS,$$

where k is an arbitrary national number. If the function λ decreases rapidly enough as $|x|, |y| \to \infty$, then Eq. (6.1) has infinite conserved quantities

$$E_k = \int_{R^2} \lambda^k dxdy.$$

Thus, for the Lax equation

$$L_{1,t} = [L_1, L_2],$$

$\lambda = \lambda(y, t)$ is not the eigenvalues of the operator L_1, but the integrals E_k of powers which are preserved. This property is also easyily seen from Eqs. (4.8) and (4.9). Though the Lax pair (4.8) is explicitly related with nonisospectral parameter λ, but the right of Eq. (4.9) implies that the Lax equation (4.9) is independent of the parameter λ. The above analysis inspires us to construct local conservation laws of breaking soliton equation by virtue of the Lax equation (4.9), not the Lax pair (4.8).

We introduce a new potential function

$$q_{2x} = \eta + \varepsilon\eta_x + \varepsilon^2\eta^2, \tag{6.2}$$

where ε is a constant parameter. Substituting (6.2) into the Lax equation (4.9) leads to

$$0 = L_{1,t} - f(L_1) - [L_1, L_2] = (1 + \varepsilon\partial_x + 2\varepsilon^2\eta)[\eta_t - 4(\eta + \varepsilon^2\eta^2)\eta_y - 2(q_x - \varepsilon\eta)_y\eta_x + \eta_{2x,y}],$$

which implies that $u = -q_x$ given by (6.2) is a solution of breaking soliton equation (1.1) if η satisfies the following equation

$$\eta_t - 4(\eta + \varepsilon^2\eta^2)\eta_y - 2(q_x - \varepsilon\eta)_y\eta_x + \eta_{2x,y} = 0. \tag{6.3}$$

However, it follows from (6.2) that

$$[(q_x - \varepsilon\eta)_y]_x = (\eta + \varepsilon^2\eta^2)_y,$$

on account of which, Eq. (6.3) is then written in a divergent-type form

$$\eta_t + [2\eta(\varepsilon^2\eta - q_x)_y]_x + (\eta_{2x} - \eta^2)_y = 0. \tag{6.4}$$

To proceed, inserting the expansion

$$\eta = \sum_{n=0}^{\infty} I_n(q, q_x, q_y \cdots)\varepsilon^n, \tag{6.5}$$

into Eq. (6.2) and equating the coefficients for power of ε, we obtain the recursion relations for I_n

$$I_0 = q_{2x} = -u_x, \quad I_1 = -I_{0,x} = u_{2x},$$
$$I_n = -I_{n-1,x} - \sum_{k=0}^{n-2} I_k I_{n-2-k}, \quad n = 3, 4, \ldots, \tag{6.6}$$

Again substituting (6.5) into (6.4) and comparing the power of ε provides us infinite consequence of conservation laws

$$I_{n,t} + F_{n,x} + G_{n,y} = 0, \; n = 1, 2, \ldots. \tag{6.7}$$

In Eq. (6.7), the conversed densities $I_n' s$ are given by formula (6.4), while the first fluxes $F_n' s$ are given by recursion formulas explicitly

$$F_0 = -2u_x u_y, \quad F_1 = 2u_{2x}u_y + 2u_x u_{xy},$$
$$F_n = 2u_y I_n + 2\sum_{k=0}^{n-1} I_k I_{n-1-k,y}, \quad n = 2, 3, \cdots \tag{6.8}$$

and the second fluxes $G_n' s$ are

$$G_0 = -u_{3x} - u_x^2, \quad G_1 = u_{4x} + 2u_x u_{2x},$$
$$G_n = I_{n,xx} - \sum_{k=0}^{n} I_k I_{n-k,y}, \quad n = 1, 2, \cdots. \tag{6.9}$$

We present recursion formulas (6.6), (6.8), and (6.9) for generating an infinite sequence of local conservation laws (6.7), the first few conserved densities and associated fluxes are explicitly given. The first equation of conservation law Eq. (6.7) is exactly the breaking soliton equation (1.1). In conclusion, the breaking soliton equation (1.1) is completely integrable in the sense that it admits bilinear Bäcklund transformation, Lax pair, and infinitely local conservation laws.

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